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2002 J. Phys. A: Math. Gen. 35 4729

(http://iopscience.iop.org/0305-4470/35/22/305)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4729–4736 PII: S0305-4470(02)33678-3

Equivalence of Dirac formulations

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Received 5 February 2002, in final form 12 March 2002
Published 24 May 2002
Online at stacks.iop.org/JPhysA/35/4729

Abstract
We construct general Dirac theories in both $\mathbb{C}\otimes\mathcal{C}\ell(3,1)$ and $\mathbb{C}\otimes\mathcal{C}\ell(1,3)$ using a first order left acting Dirac operator. Any two such theories are equivalent provided they have the same dimension. We also show that every 16- or 8-dimensional real Dirac theory in $\mathcal{C}\ell(1,3)$ is equivalent to some (complex) Dirac theory in $\mathbb{C}\otimes\mathcal{C}\ell(1,3)$. As an immediate consequence of this we have that the Hestenes and original Dirac formulations are equivalent.

PACS numbers: 03.50.Dc, 03.30.+p

1. Introduction

Dirac's original equation was constructed to fulfil three requirements: Lorentz covariance, a positive-definite conserved current and Einstein's mass–energy relation (that is, the Klein–Gordon equation). We re-examine Dirac's original argument to construct Dirac formulations for a left acting first order Dirac operator over a suitable Lorentz invariant subspace of $\mathbb{C}\otimes\mathcal{C}\ell(\eta)$. Here $\mathbb{C}\otimes\mathcal{C}\ell(\eta)$ represents the Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell(2+\eta,2-\eta)$ where $\eta=\pm1$. These include Dirac's original matrix formulation, or more precisely its reformulation in Lounesto [1] as the equation $\mathrm{i}\gamma_\mu\partial^\mu\psi=m\psi$ over the four-dimensional left ideal $\mathbb{C}\otimes\mathcal{C}\ell(1,3)\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+\mathrm{i}\gamma_{12})$. We have little interest in the 4×4 matrix algebra generated by Dirac's matrices, and work instead with the Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell(\eta)$ generated by $\mathbf{e}_\mu$. One may obtain Dirac's matrix algebra by identifying $\mathbf{e}_\mu$ and γ_μ. We refer to Dirac's original formulation in $\mathbb{C}\otimes\mathcal{C}\ell(\eta)$ as the Lounesto formulation.

The Dirac formulations given by Joyce [2] utilize the Dirac equation $\mathrm{i}\nabla\psi=m\psi\mathbf{e}_0$ over the subalgebra $\mathbb{C}\otimes\mathcal{C}\ell^+(\eta)$, where $\nabla=\mathbf{e}_\mu\partial^\mu$. This is equivalent to two copies of the Lounesto formulation, because of a right acting $SU(2)$ gauge freedom. We may fix the gauge by restricting to $\mathbb{C}\otimes\mathcal{C}\ell^+(1,3)\frac{1}{2}(1+\mathrm{i}\mathbf{e}_{12})$ which results in equivalence with Lounesto's formulation. In general we demonstrate that two Dirac formulations whose fibre spaces are of the same dimension 4, 8, 12 or 16 are equivalent. Furthermore, we show that this carries over to real Dirac formulations of (real) dimension 8 and 16 in the Clifford algebra $\mathcal{C}\ell(1,3)$. This includes Hestenes' formulation where the Dirac equation is $\nabla\psi=m\psi\mathbf{e}_{012}$ over the subalgebra $\mathcal{C}\ell^+(1,3)$.

The geometry of spacetime is represented by either of the non-isomorphic Clifford algebras $\mathcal{C}\ell(3,1)$ or $\mathcal{C}\ell(1,3)$. It is argued by Hestenes [3] and Gull *et al* [4], to name a few, that there are many copies of $\mathbb{C}$ in both these algebras and hence there is no need to complexify. In particular we have $\mathcal{C}\ell^+(3,1) \cong \mathcal{C}\ell^+(1,3) \cong \mathcal{C}\ell(3,0) \cong \mathbb{C}\otimes\mathbb{H}$. Nevertheless, traditional Dirac theory is formulated in the Dirac algebra. There are three different ways to view the Dirac algebra $\mathbb{C}\otimes\mathcal{C}\ell(3,1) \cong \mathbb{C}\otimes\mathcal{C}\ell(1,3)$. Firstly, and traditionally, as an abstract space acting on the space of spinors. Secondly, as the fibre space of a complex geometric field, where each multivector is assigned a phase factor. Thirdly, as in Kaluza–Klein theory where we introduce a fourth spatial direction, since we have $\mathcal{C}\ell(4,1) \cong \mathbb{C}\otimes\mathcal{C}\ell(3,1) \cong \mathbb{C}\otimes\mathcal{C}\ell(1,3) \cong \mathcal{C}\ell(1,4)$.

2. Formulation of Dirac theories

Dirac theories arise from first order differential equations covariant under the Poincaré group. The Dirac field is a smooth map $\psi : \mathcal{M}^4 \to \mathbb{C}\otimes\mathcal{C}\ell(\eta)$ where $\mathcal{M}^4$ is Minkowski spacetime. We note that when we write $\psi \in \mathbb{C}\otimes\mathcal{C}\ell(\eta)$ we mean $\psi(\mathbf{x}) \in \mathbb{C}\otimes\mathcal{C}\ell(\eta)$ for all $\mathbf{x} \in \mathcal{M}^4$. Indulging in such notational abuse requires that we remain aware of the distinction between a Dirac field, which is a map $\psi : \mathcal{M}^4 \to \mathbb{C}\otimes\mathcal{C}\ell(\eta)$, and the value $\psi(\mathbf{x})$ the field attains at $\mathbf{x}$.

An infinitesimal analysis of translations of $\mathcal{M}^4$ for any Dirac field ψ extracts the energy operator as $E = -\mathrm{i}\partial^0$ and the kth component of the momentum operator as $P^k = \mathrm{i}\partial^k$. We define the Dirac operator to be $\nabla = \mathbf{e}_\mu\partial^\mu$ which in terms of the energy–momentum operators is $\nabla = \mathrm{i}\mathbf{e}_0 E - \mathrm{i}\mathbf{e}_k P^k$. Einstein's mass–energy relationship (or the first Casimir invariant) implies the Klein–Gordon equation

$$\nabla^2\psi^2 = \eta m^2\psi \tag{1}$$

where $\nabla^2 = \eta(E^2 - P^2)$. Dirac theories seek to factorize the Klein–Gordon operator $\eta\nabla^2 - m^2$ to obtain a first order differential equation with a conserved energy–momentum current. We seek an equation of the form

$$\mathbf{D}\psi = m\psi\mathbf{k} \tag{2}$$

where $\mathbf{D} = \mathbf{a}_\mu\partial^\mu$. The Clifford elements $\mathbf{a}_\mu, \mathbf{k} \in \mathbb{C}\otimes\mathcal{C}\ell(\eta)$ are required by translational covariance to be constant over the base space $\mathcal{M}^4$. Applying $\mathbf{D}$ to the equation gives $\mathbf{D}^2\psi = m^2\psi\mathbf{k}^2$. Einstein's mass–energy relation requires $\mathbf{D}^2 = c\nabla^2$ and $\mathbf{k}^2 = c\eta$ for some complex number $c \neq 0$. The first condition requires the relationship

$$\{\mathbf{a}_\mu, \mathbf{a}_\nu\} = c\{\mathbf{e}_\mu, \mathbf{e}_\nu\}. \tag{3}$$

The collection $\{\mathbf{a}_\mu\}$ forms a generating set for $\mathbb{C}\otimes\mathcal{C}\ell(\eta)$. We assume that each $\mathbf{a}_\mu$ is odd. Thus there is an invertible element $\mathbf{R}$ such that $\mathbf{a}_\mu = \sqrt{c}\,\mathbf{R}^{-1}\mathbf{e}_\mu\mathbf{R}$ or $\mathbf{a}_\mu = \eta\sqrt{c}\,\mathbf{e}_{0123}\mathbf{R}^{-1}\mathbf{e}_\mu\mathbf{R}$. This gives respectively $D = \sqrt{c}\,\mathbf{R}^{-1}\nabla\mathbf{R}$ and $\mathbf{D} = \eta\sqrt{c}\,\mathbf{e}_{0123}\mathbf{R}^{-1}\nabla\mathbf{R}$. In the former case we take $\mathbf{R}^{-1}\mathbf{e}_\mu\mathbf{R}$, and in the latter case $|c|^{-1/2}\mathbf{e}_{0123}\mathbf{R}^{-1}\mathbf{e}_\mu\mathbf{R}$, as the generators $\mathbf{e}_\mu$ of the Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell(\eta)$. Under this transformation we can transform $\mathbf{D}$ into ∇ (and $\mathbf{K}$ into $\mathbf{k}$) giving the general form of the Dirac equation as

$$\nabla\psi = m\psi\mathbf{k} \tag{4}$$

with $\mathbf{k}^2 = \eta$. We also require the Dirac equation to have a conserved probability current. The unique inner product with positive definite norm is given by $\langle\bar{\psi}\psi\rangle_0$ where the Dirac conjugate of an n-vector $\mathbf{v}$ is given by $\bar{\mathbf{v}} = (-\eta)^n\mathbf{e}_0\mathbf{v}^\dagger\mathbf{e}_0$. From this we construct the Dirac current

$$J_\mu = \left\langle\bar{\psi}\mathbf{e}_0^{-1}\mathbf{e}_\mu\psi\right\rangle_0. \tag{5}$$

Taking the divergence, inserting the Dirac equation and using the cyclic properties of the trace shows that $\partial^\mu J_\mu = m\left\langle\left(\bar{\mathbf{k}}\mathbf{e}_0^{-1} + \mathbf{k}\mathbf{e}_0\right)\bar{\psi}\psi\right\rangle_0$. Thus the current is conserved whenever $\bar{\mathbf{k}} = \eta\mathbf{k}$. Since $\mathbf{k}$ characterizes the nature of the Dirac equation we make the following definition.

Definition 1. *If* $\mathbf{k} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ *satisfies* $\mathbf{k}^2 = \eta$ *and* $\bar{\mathbf{k}} = \eta\mathbf{k}$ *then we call* $\mathbf{k}$ *a Dirac character.*

The Clifford elements $\mathbf{k}$ satisfying $\bar{\mathbf{k}} = \eta\mathbf{k}$ comprise a real linear subspace of $\mathbb{C} \otimes \mathcal{C}\ell(\eta)$ spanned by $\sqrt{\eta}$, $\mathrm{i}\mathbf{e}_0$, $\mathbf{e}_k$, $\sqrt{\eta}\mathbf{e}_{0k}$, $\sqrt{-\eta}\mathbf{e}_{kl}$, $\mathbf{e}_{0kl}$, $\mathrm{i}\mathbf{e}_{123}$ and $\sqrt{-\eta}\mathbf{e}_{0123}$. These basis elements all square to η and hence are Dirac characters. In the literature there are a number of choices for $\mathbf{k}$. For Dirac's original equation $\eta = -1$ and $\mathbf{k} = -\mathrm{i}$, for Hestenes' equation [5] $\mathbf{k} = \mathbf{e}_{012}$ and for the equation of Joyce [2] $\mathbf{k} = -\mathrm{i}\mathbf{e}_0$.

The *physical content* of the Dirac field ψ is represented by the quantum numbers for energy, momentum and angular momentum (including intrinsic spin). The transformations for translation, rotation and boost are all of the form $\psi \mapsto \mathbf{U}\psi(\mathbf{U}^{-1}\mathbf{x}\mathbf{U})$. An infinitesimal analysis shows that the operators corresponding to the above observables are even Clifford operators and hence commute with $\mathbf{e}_{0123}$. Thus a transformation $\psi \mapsto \mathbf{e}_{0123}\psi$ changes the sign of the mass but not the physical content of the Dirac field. More generally, we define a gauge transformation of the Dirac equation to be any transformation preserving the physical content of the Dirac field. These transformations direct sum decompose into two subspaces depending on whether or not they commute with $\mathbf{k}$. Suppose $\mathbf{U}$ is an invertible Clifford element. If $[\mathbf{k}, \mathbf{U}] = 0$, then ψ transforms according to the right action

$$\psi \mapsto \psi\mathbf{U} \tag{6}$$

and if $\{\mathbf{k}, \mathbf{U}\} = 0$, then ψ transforms according to the pseudo-right action

$$\psi \mapsto \mathbf{e}_{0123}\psi\mathbf{U}. \tag{7}$$

A companion paper [6] analyses the consequences of determining the full compact gauge group resulting from these transformations.

Dirac theories invariably restrict the image of the Dirac field ψ to some linear subspace $\mathcal{J}$ of $\mathbb{C} \otimes \mathcal{C}\ell(\eta)$. This is equivalent to fixing the gauge under the pseudo-right/right gauge actions. Furthermore, the subspace $\mathcal{J}$ must be invariant under the Lorentz group. This requires that $\mathbf{u}\psi \in \mathcal{J}$ for all $\mathbf{u} \in \mathbb{C} \otimes \mathcal{C}\ell^+(\eta)$ and all $\psi \in \mathcal{J}$. That is, $\mathbb{C} \otimes \mathcal{C}\ell^+(\eta)\mathcal{J} = \mathcal{J}$. We call such a subspace $\mathcal{J}$ a pseudo-left ideal. The even subalgebra $\mathbb{C} \otimes \mathcal{C}\ell^+(\eta)$ of $\mathbb{C} \otimes \mathcal{C}\ell(\eta)$ is an example of a pseudo-left ideal that is not a left ideal, as is the odd linear subspace $\mathbb{C} \otimes \mathcal{C}\ell^-(\eta)$. Let $\mathbf{k} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ be such that $\mathbf{k}^2 = -1$ and define $\mathbf{T}\psi = \mathbf{e}_{0123}\psi\mathbf{l}$. We see that $\mathbf{T}^2 = 1$ and that $\frac{1}{2}(1 + \mathbf{T})$ are orthogonal projection operators. The subspaces $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\frac{1}{2}(1 \pm \mathbf{T})$ are pseudo-left ideals. For $\mathbf{l} = \pm\mathrm{i}$ we get the left acting chirality operators.

Given a Dirac equation with character $\mathbf{k}$ there may be no non-trivial solution whose image lies entirely within a given pseudo-left ideal. If $\psi \in \mathcal{J}$ is a solution then $\psi\mathbf{k} = \frac{1}{m}\nabla\psi \in \mathbb{C} \otimes \mathcal{C}\ell^-(\eta)\mathcal{J}$. This is a necessary condition for a non-trivial solution. Thus we make the following definition.

Definition 2. *A pseudo-left ideal* $\mathcal{J}$ *is called a Dirac pseudo-left ideal of Dirac character* $\mathbf{k}$ *if* $\mathbb{C} \otimes \mathcal{C}\ell^-(\eta)\mathcal{J} = \mathcal{J}\mathbf{k}$.

If $\mathcal{J}$ is a left ideal then it is Dirac if and only if $\mathcal{J} = \mathcal{J}\mathbf{k}$. We are now in a position to define precisely what we mean by a Dirac formulation.

Definition 3. *A Dirac formulation is a pair* $(\mathbf{k}, \mathcal{J})$ *where* $\mathbf{k}$ *is a Dirac character and* $\mathcal{J}$ *is a Dirac pseudo-left ideal.*

A Dirac formulation defines a current-conserving Dirac equation of character $\mathbf{k}$ over a suitable Lorentz invariant subspace $\mathcal{J}$. We devote the remaining sections to revealing the equivalence of Dirac formulations.

We have restricted ourselves to Dirac formulations where ∇ operates from the left. We could equally well have dealt with Dirac equations of the form $\psi\tilde{\nabla} = m\mathbf{k}\psi$ over some pseudo-right ideal where $\tilde{\nabla}$ acts from the right. Equations of this form are called adjoint Dirac equations. Moreover, reversion takes a Dirac equation $\nabla\psi = m\psi\mathbf{k}$ over $\mathcal{J}$ to an adjoint Dirac equation $\tilde{\psi}\tilde{\nabla} = m\tilde{\mathbf{k}}\tilde{\psi}$ over the pseudo-right ideal $\tilde{\mathcal{J}}$. The action of the Lorentz group on spacetime is preserved and the Dirac field transforms as $\psi \mapsto \psi\tilde{\mathbf{U}}$. Consequently the energy and momentum operators remain unaffected and the spin operators act from the right as $\tilde{\mathbf{S}}_k$. Importantly, the physical content of the Dirac field remains unchanged and the Dirac current is conserved.

3. Pseudo-left ideals

In this section we explore the properties of pseudo-left ideals. We begin with the familiar left ideals. The building blocks of left ideals are the simple (or minimal) left ideals. A simple left ideal is one that has no proper left sub-ideal. Every subspace $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ for any $\mathbf{f} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ is a left ideal. If $\mathbf{f}$ is invertible it is the entire space $\mathbb{C} \otimes \mathcal{C}\ell(\eta)$. More importantly we have the following lemma.

Lemma 1. *Every left ideal is given by* $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ *where* $\mathbf{f} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ *is idempotent.*

(Recall that $\mathbf{f}$ is idempotent if $\mathbf{f}^2 = \mathbf{f}$.) Following Lounesto [1] let $\mathrm{Mat}(4, \mathbb{C})$ denote the algebra of all 4×4 complex matrices. Recall that a matrix representation of $\mathbb{C} \otimes \mathcal{C}\ell(\eta)$ in $\mathrm{Mat}(4, \mathbb{C})$ is a 4-tuple (γ_μ) of matrices from $\mathrm{Mat}(4, \mathbb{C})$ satisfying $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ where $(g_{\mu\nu}) = \mathrm{diag}\{-\eta, \eta, \eta, \eta\}$. We prove our proposition by demonstrating that every left ideal of $\mathrm{Mat}(4, \mathbb{C})$ may be written as $\mathrm{Mat}(4, \mathbb{C})f$, where $f \in \mathrm{Mat}(4, \mathbb{C})$ is idempotent. Suppose $\mathcal{J}$ is a left ideal of $\mathrm{Mat}(4, \mathbb{C})$. Fix $A \in \mathcal{J}$ satisfying $\mathrm{rank}\,(A) = \max_{A' \in \mathcal{J}} \mathrm{rank}\,(A')$. We know from elementary linear algebra that any $m \times n$ matrix A may be factored as $A = LI_rR$. Here L (respectively R) is a non-singular $m \times m$ (respectively $n \times n$) matrix, and the $m \times n$ matrix I_r consists of zeros everywhere except in the first r diagonal positions, where the number of '1' entries is $r = \mathrm{rank}\,(A)$. If $\mathcal{J}$ is a left ideal of $\mathrm{Mat}(n, \mathbb{C})$ and $A \in \mathcal{J}$ then $R^{-1}I_rR \in \mathcal{J}$. We now change to the representation (γ'_μ) via the similarity transformation $R\gamma_\mu R^{-1}$. In this new representation $(R^{-1}I_rR)' = I_r \in \mathcal{J}'$. Thus $\mathcal{J}'$ contains all 4×4 complex matrices with the last $4 - r$ columns containing only zero entries. That is, $\mathrm{Mat}(4, \mathbb{C})I_r \subset \mathcal{J}'$. We now show that all $B \in \mathcal{J}'$ are of this form. Since $BI_r \in \mathcal{J}'$ then $B(1 - I_r) \in \mathcal{J}'$. The entries of the first r columns of $B(1 - I_r)$ are all zero. We will show that $B(1 - I_r) = 0$. If $(1 - I_r)B(1 - I_r) \neq 0$ then $I_r + B(1 - I_r) \in \mathcal{J}'$ whose rank exceeds that of A contradicting the choice of A. Hence $(1 - I_r)B(1 - I_r) = 0$. Suppose b_{ij} is a nonzero entry of $I_rB(1 - I_r)$ then $j > r \geqslant i$ and a row interchange, $R_{i,r+1}$, of the ith and $(r + 1)$th rows of $I_rB(1 - I_r)$ shifts b_{ij} into the $(r + 1)$th row, jth column of $R_{i,r+1}I_rB(1 - I_r) \in \mathcal{J}'$. This contradicts the requirement that $(1 - I_r)R_{i,r+1}I_rB(1 - I_r) = 0$. Thus $B(1 - I_r) = 0$ showing $\mathcal{J}' \subset \mathrm{Mat}(4, \mathbb{C})I_r$. That is, $\mathcal{J}' = \mathrm{Mat}(4, \mathbb{C})I_r$ and the left ideal represented by $\mathcal{J}$ in $\mathbb{C} \otimes \mathcal{C}\ell(\eta)$ is given by $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ where $\mathbf{f} = \sum_{\alpha \in I} u^\alpha \mathbf{e}_\alpha$ and the $u^\alpha \in \mathbb{C}$ are given by the expansion of I_r in the matrix representation as

$$I_r = \sum_{\alpha \in I} u^\alpha \gamma'_\alpha.$$

The proof of the lemma reveals that a left ideal is simple if and only if its dimension is 4. Every left ideal is semisimple. A Gram–Schmidt orthogonalization applied to any left ideal

$\mathcal{J}$ gives the direct sum decomposition

$$\mathcal{J} = \bigoplus_{i=1}^{r} \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}_i \tag{8}$$

where each summand is simple and $\mathbf{f}_i\mathbf{f}_j = \delta_{ij}\mathbf{f}_j$. Consequently $\mathbf{f} = \sum_{i=1}^{r} \mathbf{f}_i$ is idempotent and $\mathcal{J} = \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ with $\dim \mathcal{J} = 4r$. We define the rank of $\mathbf{f}$ to be $\operatorname{rank} \mathbf{f} = r$.

Given the preceding lemma we can determine the condition that a left ideal is a Dirac left ideal.

Proposition 1. *A left ideal $\mathcal{J} = \mathbb{C} \otimes \mathcal{C}\ell(\eta)\frac{1}{2}(1 + \mathbf{l})$ is a Dirac left ideal of character $\mathbf{k}$ if and only if $[\mathbf{k}, \mathbf{l}]\frac{1}{2} \in \mathcal{J}$.*

The left ideal is a Dirac left ideal of character $\mathbf{k}$ if and only if $\mathbf{k}'(1 + \mathbf{l}) = (1 + \mathbf{l})\mathbf{k}$ for some $\mathbf{k}' \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$. We have that $(1 + \mathbf{l})\mathbf{k} = (\mathbf{k} + \mathbf{f}\mathbf{k})\frac{1}{2}(1 + \mathbf{l}) + (\mathbf{k} + \mathbf{l}\mathbf{k})\frac{1}{2}(1 - \mathbf{l})$ so the left ideal is a Dirac ideal of character $\mathbf{k}$ if and only if the second term vanishes. This term can be rewritten as $(\mathbf{k}(1 + \mathbf{f}) - [\mathbf{k}, \mathbf{l}])\frac{1}{2}(1 - \mathbf{l})$ and equals zero if and only if $[\mathbf{k}, \mathbf{l}]\frac{1}{2}(1 - \mathbf{l}) = 0$.

The general subspaces of interest are the pseudo-left ideals. This class contains the left ideals. We define a simple pseudo-left ideal to be one that has no proper left sub-ideal. It is important to realize that a left ideal cannot be a simple pseudo-left ideal even if it is itself a simple left ideal. In fact every left ideal $\mathcal{J}$ can be decomposed into two pseudo-left ideals $\frac{1}{2}(1 \pm \mathrm{i}\mathbf{e}_{0123})\mathcal{J}$ using the chirality projectors. The subspaces $\mathbb{C} \otimes \mathcal{C}\ell^{\pm}(\eta)\mathbf{f}$ are pseudo-left ideals for any $\mathbf{f} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$. A pseudo-left ideal $\mathcal{J}$ is called strict if $\mathcal{J} \cap (\mathbb{C} \otimes \mathcal{C}\ell^{-}(\eta)\mathcal{J}) = \{0\}$. In this case we see that $\mathbb{C} \otimes \mathcal{C}\ell^{-}(\eta)\mathcal{J}$ is also a strict pseudo-left ideal with the same dimension as $\mathcal{J}$. Moreover, the direct sum $\mathcal{J} \oplus (\mathbb{C} \otimes \mathcal{C}\ell^{-}(\eta)\mathcal{J})$ is a left ideal. The dimension of all strict pseudo-left ideals must be half that of some left ideal. That is, of dimension 8, 6, 4 or 2. Examples for each dimension are $\mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)$, $\mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\left(1 - \frac{1}{4}(1 + \mathrm{i}\mathbf{e}_{12})(1 + \mathbf{e}_{03})\right)$, $\mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\frac{1}{2}(1 + \mathrm{i}\mathbf{e}_{12})$ and $\mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\frac{1}{4}(1 + \mathrm{i}\mathbf{e}_{12})(1 + \mathbf{e}_{03})$, respectively.

Every pseudo-left ideal can be written as the direct sum of a left ideal and a strict pseudo-left ideal as attested by the following proposition. Define the right acting gauge transformation $T_{\pm}$ by $\psi T_{\pm} = \frac{1}{2}(\psi \mp \mathbf{e}_{0123}\psi\mathbf{e}_{0123})$. Hence $\mathbb{C} \otimes \mathcal{C}\ell(\eta)T_{\pm} = \mathbb{C} \otimes \mathcal{C}\ell^{\pm}(\eta)$.

Proposition 2. *Every pseudo-left ideal is of the form*

$$\mathbb{C} \otimes \mathcal{C}\ell(\eta)(\mathbf{f} + T_{+}\mathbf{h}\mathbf{g}) = (\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}) \oplus (\mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\mathbf{h}\mathbf{g}) \tag{9}$$

where $\mathbf{f}$ and $\mathbf{g}$ are orthogonal idempotents, $\mathbf{h}$ is invertible and $(\mathbf{h}T_{-})\mathbf{g} \neq 0$ unless $\mathbf{g} = 0$.

Let $\mathcal{J}$ be a pseudo-left ideal and $\mathcal{J}_1$ a maximal left ideal contained in $\mathcal{J}$. There exists $\mathbf{f}$ idempotent such that $\mathcal{J}_1 = \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$. Let $\mathcal{J}_2 \subset \mathcal{J}_1^{\perp} = \mathbb{C} \otimes \mathcal{C}\ell(\eta)(1 - \mathbf{f})$ such that $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$, then $\mathcal{J}_2$ is a strict pseudo-left ideal. Now $(\mathbb{C} \otimes \mathcal{C}\ell^{-}(\eta)\mathcal{J}_2) \oplus \mathcal{J}_2$ is a left ideal given by $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{g}$ for some idempotent $\mathbf{g}$ orthogonal to $\mathbf{f}$. Moreover, there exists an invertible $\mathbf{h}$ such that $\mathbf{h}\mathbf{g} \in \mathcal{J}_2$. Hence $\mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\mathbf{h}\mathbf{g} \subset \mathcal{J}_2$. Dimension counting reveals that $\dim \mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\mathbf{h}\mathbf{g} = \dim \mathcal{J}_2$ showing $\mathcal{J}_2 = \mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\mathbf{h}\mathbf{g}$. Finally, if $(\mathbf{h}T_{-})\mathbf{g} = 0$ then there is $\mathbf{u} \in \mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)$ such that $\mathbf{u}\mathbf{g} = \mathbf{h}\mathbf{g}$. Since $\mathbf{u}$ commutes with T_{+} then it can be absorbed and we may choose $\mathbf{h} = 1$.

We note that every pseudo-left ideal $\mathbb{C} \otimes \mathcal{C}\ell(\eta)(\mathbf{f} + T_{+}\mathbf{h}\mathbf{g})$ of the above proposition is gauge equivalent to a pseudo-left ideal $\mathbb{C} \otimes \mathcal{C}\ell(\eta)(\mathbf{f}' + T_{+}\mathbf{g}')$ where $\mathbf{f}' = \mathbf{h}\mathbf{f}\mathbf{h}^{-1}$ and $\mathbf{g}' = \mathbf{h}\mathbf{g}\mathbf{h}^{-1}$ are orthogonal idempotents, and the gauge equivalence is $\psi \mapsto \psi\mathbf{h}^{-1}$. A Gram–Schmidt orthogonalization shows that every strict pseudo-left ideal $\mathcal{J}$ is semisimple. Hence we have the semisimple decomposition

$$\mathcal{J} = \bigoplus_{i=1}^{r} \mathbb{C} \otimes \mathcal{C}\ell^{+}(\eta)\mathbf{h}_i\mathbf{f}_i \tag{10}$$

where the $\mathbf{h}_i$ are invertible and $\mathbf{f}_i\mathbf{f}_j = \delta_{ij}\mathbf{f}_j$.

We have the condition of the following proposition for pseudo-left ideals to be Dirac.

Proposition 3. *A pseudo-left ideal $\mathbb{C} \otimes \mathcal{C}\ell(\eta)(\mathbf{f} + T_{+}\mathbf{hg})$ where $\mathbf{h}$ is invertible and $\mathbf{f}$ and $\mathbf{g}$ are orthogonal idempotents is Dirac of character $\mathbf{k}$ if and only if there are $\mathbf{u}, \mathbf{v} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ invertible such that $\mathbf{v}$ is odd, $\mathbf{fk} = \mathbf{uf}$ and $\mathbf{gk} = \mathbf{vg}$.*

We first note that $\mathbb{C} \otimes \mathcal{C}\ell(\eta)(\mathbf{f} + T_{+}\mathbf{hg})$ is Dirac of character $\mathbf{k}$ if and only if for all $\psi \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)(\mathbf{f} + T_{+}\mathbf{hg})$ we have that $\psi\mathbf{f} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ if and only if $\psi\mathbf{fk} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ and that $\psi\mathbf{g} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)T_{+}\mathbf{hg}$ if and only if $\psi\mathbf{gk} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)T_{-}\mathbf{hg}$. Thus a pseudo-left ideal is Dirac of character $\mathbf{k}$ if and only if its component strict pseudo-left and left ideal components are Dirac of character $\mathbf{k}$. If $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ is Dirac of character $\mathbf{k}$ then from the definition there is an invertible $\mathbf{u} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ such that $\mathbf{fk} = \mathbf{uf}$. The converse also holds. If $\mathbb{C} \otimes \mathcal{C}\ell(\eta)T_{+}\mathbf{hg}$ is Dirac of character $\mathbf{k}$ then from the definition there is an invertible $\mathbf{v}' \in \mathbb{C} \otimes \mathcal{C}\ell^{-}(\eta)$ such that $\mathbf{hgk} = \mathbf{v}'\mathbf{hg}$. Thus $\mathbf{gk} = \mathbf{vg}$ where $\mathbf{v} = \mathbf{h}^{-1}\mathbf{v}'\mathbf{h} \in \mathbb{C} \otimes \mathcal{C}\ell^{-}(\eta)$ and is invertible. The converse also holds.

Finally note that the proposition implies the Dirac left ideal condition of proposition 1 since $[\mathbf{hg}, \mathbf{k}] = (\mathbf{vh} - \mathbf{kh})\mathbf{g}$ which annihilates with $1 - \mathbf{g}$, and $[\mathbf{f}, \mathbf{k}] = (\mathbf{u} - \mathbf{k})\mathbf{f}$ which annihilates with $1 - \mathbf{f}$.

4. Equivalence of Dirac formulations

We begin the section by making explicit exactly what we mean when we say that two Dirac formulations are equivalent.

Definition 4. *Two Dirac formulations $(\mathbf{k}_1, \mathcal{J}_1)$ and $(\mathbf{k}_2, \mathcal{J}_2)$ are equivalent, written $(\mathbf{k}_1, \mathcal{J}_1) \cong (\mathbf{k}_2, \mathcal{J}_2)$, if there exists a gauge transformation $\phi : \mathcal{J}_1 \to \mathcal{J}_2$.*

The gauge transformation ϕ is an isomorphism of vector spaces preserving the physical content mapping Dirac fields of character $\mathbf{k}_1$ to Dirac fields of character $\mathbf{k}_2$. We begin with the case when $\mathcal{J}_1$ and $\mathcal{J}_2$ are both left ideals.

Proposition 4. *Two Dirac formulations $(\mathbf{k}_1, \mathcal{J}_1)$ and $(\mathbf{k}_2, \mathcal{J}_2)$ where $\mathcal{J}_1$ and $\mathcal{J}_2$ are left ideals are equivalent if and only if* $\dim \mathcal{J}_1 = \dim \mathcal{J}_2$.

We first demonstrate that given a Dirac formulation $(\mathbf{k}, \mathcal{J})$ where $\mathcal{J}$ is a left ideal then $(\mathbf{k}, \mathcal{J}) \cong (\sqrt{\eta}, \mathcal{J})$. We note that by proposition 1 that $\mathcal{J}$ is a Dirac left ideal of character $\sqrt{\eta}$. We also note the important identity

$$\mathbf{k}\tfrac{1}{2}(1 \pm \sqrt{\eta}\mathbf{k}) = \pm\eta\sqrt{\eta}\tfrac{1}{2}(1 \pm \sqrt{\eta}\mathbf{k}). \tag{11}$$

Define the transformation $T : \mathcal{J} \to \mathcal{J}$ by

$$T\psi = \psi\tfrac{1}{2}(1 + \sqrt{\eta}\mathbf{k}) + \mathbf{e}_{0123}\psi\tfrac{1}{2}(1 - \sqrt{\eta}\mathbf{k}). \tag{12}$$

This map is invertible with inverse $T^{-1}\psi = \psi\frac{1}{2}(1 + \sqrt{\eta}\mathbf{k}) - \mathbf{e}_{0123}\psi\frac{1}{2}(1 - \sqrt{\eta}\mathbf{k})$. Clearly T preserves physical content and by identity (11) maps a Dirac field ψ of character $\mathbf{k}$ to a Dirac field $T\psi$ of character $\eta\sqrt{\eta}$. Finally, in the case $\eta = -1$ apply the transformation $\psi \mapsto \mathbf{e}_{0123}\psi$, we obtain the desired equivalence. To complete the proof of the proposition we show that the

Dirac formulations $(\sqrt{\eta}, \mathcal{J}_1)$ and $(\sqrt{\eta}, \mathcal{J}_2)$ are equivalent if and only if $\dim \mathcal{J}_1 = \dim \mathcal{J}_2$. Let $\mathcal{J}_i = \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}_i$ where $\mathbf{f}_i$ are idempotent and $i = 1, 2$. Embedded in the proof of lemma 1 we demonstrated that there exist invertibles $\mathbf{u}, \mathbf{v} \in \mathbb{C} \otimes \mathcal{C}\ell(\eta)$ such that $\mathbf{f}_2 = \mathbf{v}\mathbf{f}_1\mathbf{u}$. We define $T : \mathcal{J}_1 \to \mathcal{J}_2$ by $T\psi = \psi\mathbf{u}$ then it is easy to check that this is an equivalence.

There are no Dirac formulations $(\mathbf{k}, \mathcal{J})$ where $\mathcal{J}$ is a strict pseudo-left ideal and $\mathbf{k}$ is even. This includes the choices $\mathbf{k} = \sqrt{\eta}$, showing in particular that Dirac's original equation over any strict pseudo-left ideal has only trivial solutions. Nevertheless, every Dirac formulation over a strict pseudo-left ideal of dimension 4 or 8 is equivalent to a Dirac formulation over a left ideal, as we now demonstrate.

Proposition 5. *Given a Dirac formulation* $(\mathbf{k}, \mathbb{C} \otimes \mathcal{C}\ell^+(\eta)\mathbf{hg})$ *over a strict pseudo-left ideal with* $\operatorname{rank} \mathbf{g}$ *of* 2 *or* 4*, then there is an equivalent Dirac formulation* $(\mathbf{k}', \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f})$.

We have that $(\mathbf{k}, \mathbb{C} \otimes \mathcal{C}\ell^+(\eta)\mathbf{hg}) \cong (\mathbf{hkh}^{-1}, \mathbb{C} \otimes \mathcal{C}\ell^+\mathbf{g}')$ where $\mathbf{g}' = \mathbf{hgh}^{-1}$ is idempotent under the gauge equivalence $\psi \mapsto \psi\mathbf{h}^{-1}$. Thus without loss of generality we assume that $\mathbf{h} = 1$. Define $\mathbf{f} = \mathbf{g}\frac{1}{2}(1 + \sqrt{\eta}\mathbf{k})$. There exists $\mathbf{v} \in \mathbb{C} \otimes \mathcal{C}\ell^-(\eta)$ invertible such that $\mathbf{gk} = \mathbf{vg}$. Then $\mathbf{f} = \frac{1}{2}(1 + \sqrt{\eta}\mathbf{v})\mathbf{g}$ and is idempotent. We have that $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f} \subset \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{g}$ where either $\operatorname{rank} f = \operatorname{rank} g$ or $\operatorname{rank} \mathbf{f} = \frac{1}{2}\operatorname{rank} \mathbf{g}$. The former case gives $\mathbf{f} = \mathbf{g} = \frac{1}{2}(1 + \sqrt{\eta}\mathbf{k}')$ for some $\mathbf{k}'$ squaring to η. Since $\mathbf{vf} = \eta\sqrt{\eta}\mathbf{f}$ then $\mathbb{C} \otimes \mathcal{C}\ell^+(\eta)\mathbf{f} = \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f}$ which is a left ideal contrary to the hypothesis of the proposition. Hence $\operatorname{rank} \mathbf{f} = \frac{1}{2}\operatorname{rank} \mathbf{g}$ and $\mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{g} = \mathbb{C} \otimes \mathcal{C}\ell^+(\eta)\mathbf{g} \oplus \mathbb{C} \otimes \mathcal{C}\ell^-(\eta)\mathbf{g}$. We define the canonical projection operators $P_\pm : \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{g} \to \mathbb{C} \otimes \mathcal{C}\ell^\pm(\eta)\mathbf{g}$. These satisfy $\mathbf{u}P_\pm = P_\pm\mathbf{u}$ for all $\mathbf{u} \in \mathbb{C} \otimes \mathcal{C}\ell^+$. We have $(\mathbf{k}, \mathbb{C} \otimes \mathcal{C}\ell^+(\eta)\mathbf{g}) \cong (\mathbf{k}, \mathbb{C} \otimes \mathcal{C}\ell(\eta)\mathbf{f})$ under the gauge equivalence $\psi \mapsto \psi\frac{1}{2}(1 + \sqrt{\eta}\mathbf{k})$, whose inverse is $\psi \mapsto P_+\psi$.

5. Real Dirac formulations

The first formulation of a real Dirac theory was that of Hestenes [5, 7]. The extraction of observables is more delicate with real Dirac formulations. This is because no element of $\mathcal{C}\ell(\eta)$ both squares to -1 and commutes with every other element. Hestenes' way around this problem is to extract the quantum numbers from the Dirac current. Alternatively, the unit imaginary i may be identified with $\mathbf{e}_{0123}$ giving the spin operators as $\mathbf{S}_k = \frac{1}{2}\mathbf{e}_{0k}$. Whatever the interpretation is we carry the definitions directly over from the complex situation to the real situation. This gives us the following proposition.

Proposition 6. *If* $\eta = -1$ *then every real Dirac formulation* $(\mathbf{k}, \mathcal{J})$ *with* $\dim \mathcal{J} = 8$ *or* 16 *is equivalent to every (complex) Dirac formulation of dimension* $\frac{1}{2}\dim \mathcal{J}$.

The real Dirac formulation is equivalent to $(\mathbf{k}, \mathbb{C} \otimes \mathcal{J}\frac{1}{2}(1 + \mathrm{i}\mathbf{k}))$ under the gauge equivalence $\psi \mapsto \psi\frac{1}{2}(1 + \mathrm{i}\mathbf{k})$ with inverse $\psi \mapsto \Re\psi$. This follows from $\mathbf{k}\frac{1}{2}(1 + \mathrm{i}\mathbf{k}) = -\mathrm{i}\frac{1}{2}(1 + \mathrm{i}\mathbf{k})$.

The above proposition provides the link to Hestenes' equation via an $SU(2)$ gauge-fixed Joyce equation. Lounesto [1] gives a matrix representation for $(-\mathrm{i}, \mathbb{C} \otimes \mathcal{C}\ell(1,3)\frac{1}{2}(1 + \mathrm{i}\mathbf{e}_0)\frac{1}{2}(1 + \mathrm{i}\mathbf{e}_{12}))$ that demonstrates its equivalence to the Lounesto formulation. The Lounesto formulation is equivalent to $(-\mathrm{i}\mathbf{e}_0, \mathbb{C} \otimes \mathcal{C}\ell^+(1,3)\frac{1}{2}(1 + \mathrm{i}\mathbf{e}_{12}))$, the Dirac equation in Joyce [2] with the $SU(2)$ gauge fixed. The gauge equivalence is given by the map $\psi \mapsto 2\psi T_+$ with inverse $\psi \mapsto \psi\frac{1}{2}(1 + \mathbf{e}_0)$. This in turn is equivalent to Hestenes' real Dirac formulation $(\mathbf{e}_{012}, \mathcal{C}\ell^+(1,3))$ under the gauge equivalence $\psi \mapsto 2\Re\psi$ with inverse $\psi \mapsto \psi\frac{1}{2}(1 + \mathrm{i}\mathbf{e}_{12})$.

6. Conclusion

We have shown that there are many ways to formulate non-trivial Dirac theories. Such theories are distinguished only by the dimension of their Dirac pseudo-left ideal. This applies to the real Dirac formulations, including that due to Hestenes. In particular we have demonstrated the equivalence of the Lounesto (Dirac's original), gauge-fixed Joyce and Hestenes' Dirac formulations:

$$\begin{aligned}\left(-\mathrm{i}, \mathbb{C}\otimes\mathcal{C}\ell(1,3)\tfrac{1}{2}(1+\mathbf{e}_0)\tfrac{1}{2}(1+\mathrm{i}\mathbf{e}_{12})\right) &\cong \left(-\mathrm{i}\mathbf{e}_0, \mathbb{C}\otimes\mathcal{C}\ell^{+}(1,3)\tfrac{1}{2}(1+\mathrm{i}\mathbf{e}_{12})\right)\\ &\cong (\mathbf{e}_{012}, \mathcal{C}\ell^{+}(1,3)).\end{aligned} \tag{13}$$

Acknowledgments

The first author would like to thank Lewis Ryder for many useful discussions and acknowledges the New Zealand Foundation for Research, Science and Technology for supporting this work (contract number UOCX0102).

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